

Quantum anchor : $U_q(sl(2))$ case

P. Akueson

ISTV, Université de Valenciennes 59304 Valenciennes, France

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Abstract

We introduce the tangent space $T(H_q)$ on the quantum hyperboloid $(\mathcal{A}_{0,q}^c)$ and equip it with an action on $\mathcal{A}_{0,q}^c$ being a deformation of the action of vectors fields on functions. An embedding $sl(2)_q \longrightarrow T(H_q)$ of q -deformed Lie algebra $sl(2)$ being an analogue of the anchor $sl(2) \longrightarrow \text{Vect}(H)$ is called “quantum anchor”.

1 Introduction

Let us consider the sphere of radius $R > 0$

$$S^2 = \{(x, y, z) \in \mathbf{K}^3; \quad x^2 + y^2 + z^2 = R^2\}$$

and the infinitesimal rotations

$$X = y\partial_z - z\partial_y, \quad Y = z\partial_x - x\partial_z, \quad Z = x\partial_y - y\partial_x.$$

(Hereafter the basic field \mathbf{K} is \mathbb{R} , but all results are still valid for its complexification. In this case we assume $\mathbf{K} = \mathbb{C}$.)

It is known that these infinitesimal rotations generate the space $\text{Vect}(S^2)$ (or $\text{Vect}(A)$, where $A = \text{Fun}(S^2)$ is the space of functions on the sphere) and $\text{Vect}(S^2)$ is the space of all vectors fields on S^2 . (In what follows all functions are assumed to be restrictions of polynomials onto the sphere in question.) This means that every element $\mathcal{X} \in \text{Vect}(S^2)$ is of the form

$$\mathcal{X} = \alpha X + \beta Y + \gamma Z; \quad \alpha, \beta, \gamma \in \text{Fun}(S^2).$$

However, the vectors fields X, Y, Z are not free. They satisfy the following identity

$$xX + yY + zZ = 0.$$

So, the tangent space $T(S^2)$ on the sphere considered as a (left) A -module can be defined by this identity, i.e., it is the factor-module of the rank 3 free module

$$M = \{\alpha X + \beta Y + \gamma Z; \quad \alpha, \beta, \gamma \in \text{Fun}(S^2)\}$$

over its submodule

$$M_1 = \{ f(xX + yY + zZ); \quad f \in \text{Fun}(S^2) \}.$$

Let us fix $k \in \mathbf{K}$ and consider a hyperboloid

$$H = \{(u, v, w) \in \mathbf{K}^3; \quad 2uw + vv + 2wu = k\}.$$

Similarly to the sphere S^2 , the space of vectors fields on the hyperboloid $\text{Vect}(H)$ (or $\text{Vect}(A)$ where $A = \text{Fun}(H)$) is generated by three infinitesimal hyperbolic rotations U, V, W satisfying the identity

$$2uW + vV + 2wU = 0. \tag{1}$$

Then, similarly to the previous case we can introduce the tangent space $T(H)$ on the hyperboloid by means of the identity (1).

We remark that in both cases the tangent modules are equipped with a Lie algebra structure and that an action

$$\beta : T(H) \otimes A \longrightarrow A, \quad A = \text{Fun}(H)$$

is well defined : apply a vector field \mathcal{X} to a function f :

$$\mathcal{X} \otimes f \longrightarrow \mathcal{X}f.$$

Thus, the tangent space $T(H)$ is realized as that of vectors fields $\text{Vect}(H)$ on H and, consequently, we have an embedding of the Lie algebra $sl(2)$ generated by the elements u, v, w into the space $\text{Vect}(H)$ generated by the elements U, V, W . Traditionally such an embedding is called an *anchor*.

Consider a quantum hyperboloid. The explicit description below, let us already give three proprieties of the corresponding “quantum function” algebra :

1. this algebra is a flat deformation of its classical counterpart (see for example [DGK] for the definition of a flat deformation),
2. its product is $U_q(sl(2))$ -covariant (that is, it is a $U_q(sl(2))$ -morphism),
3. it is in some sens q -commutative (more precisely, it is a “ q -commutative” algebra of a two parameters family of $U_q(sl(2))$ -covariant algebras considered below).

In the present note we discuss two problems : what is the “tangent space” on the quantum hyperboloid ? Do we have an analogue of the above mentioned anchor ?

First of all, we should find quantum analogues of the vectors fields U, V, W . Usually, one considers the generators X, Y, H of the quantum group (QG) $U_q(sl(2))$ (see Section 1) as such quantum analogues. However, contrary to the classical case, those generators do not satisfy any q -analogue of identity (1). So, if we introduce the tangent space on the quantum hyperboloid as the family of all linear combinations of these operators (with coefficients in the q -analogue of $\text{Fun}(H)$, called *quantum hyperboloid* in what follows), then we do not have any flat deformation of the classical tangent space $T(H)$.

Here we propose other candidates to play the role of q -analogues of the operators U, V, W , and satisfying an identity that can be viewed as a q -analogue of (1). Thus, the tangent space $T(H_q)$ on the quantum hyperboloid H_q defined as the set of all linear combinations of these quantum vectors fields modulo this identity will be a deformation with respect to its classical counterpart.

Once the tangent space is defined, the following natural problem arises : does there exist an action

$$T(H_q) \otimes A \longrightarrow A$$

which is a deformation of the classical anchor ? Here A designs the quantum analogue of the algebra $\text{Fun}(H)$. We construct such an action and give an interpretation of the elements of $T(H_q)$ as *braided vectors fields*. We will obtain an embedding of $sl(2)_q$ into $T(H_q)$, called *quantum anchor*. Here $sl(2)_q$ is the “braided” analogue of $sl(2)$ and $T(H_q)$ is generated by braided analogues of the vectors fields U, V, W .

The organization of this note is the following. In the Section 1, we recall the construction of a quantum hyperboloid. In section 2 we define the tangent space on a quantum hyperboloid and equip it with an action on all “function” on the quantum hyperboloid. Finally, we defined a *quantum anchor*.

Throughout the note, the parameter $q \in \mathbf{K}$ is assumed to be generic.

2 Quantum hyperboloid

In order to introduce a quantum hyperboloid, let us consider the QG $U_q(sl(2))$ defined, as usual, by the generators X, Y, H subject to the well known relations (cf. [CP]). Let us fix a coproduct in $U_q(sl(2))$ and the corresponding antipode and consider a spin 1 $U_q(sl(2))$ -module $V = \mathbf{V}^q$.

We only need the fact that the fusion ring of finite dimensional $U_q(sl(2))$ -modules is exactly the same as in the classical case (we consider only the finite dimensional $U_q(sl(2))$ -modules which are deformations of the classical $sl(2)$ -modules). Thus, if V_i is a spin i $U_q(sl(2))$ -module, the classical formula

$$V_i \otimes V_j = \bigoplus_{k=|i-j|}^{i+j} V_k$$

is still valid although the Clebsch-Gordan coefficients (which depend on the choice of a base) are q -deformed.

In particular, we have :

$$V^{\otimes 2} = V_0 \oplus V_1 \oplus V_2.$$

We keep the notation V for the initial space and we let V_1 be the component in $V^{\otimes 2}$ isomorphic to V . Let us fix in the spaces V, V_0, V_1 and V_2 some highest weight (h.w.) elements v, v_0, v_1 and v_2 respectively and impose the relations (which are the most general relations compatible with the action of the QG $U_q(sl(2))$) :

$$v_0 - c = 0 \quad v_1 - \hbar v = 0; \quad c, \hbar \in \mathbf{K}. \quad (2)$$

One can now deduce the complete system of equations by applying to these relations the decreasing operator $Y \in U_q(sl(2))$.

Let us denote $\mathcal{A}_{h,q}^c$ the algebra defined by the quotient algebra

$$T(V) / \{I_h\}$$

where $\{I_h\}$ is the ideal generated by the elements of the left hand side of (2) and all the derived elements.

The algebra $\mathcal{A}_{h,q}^c$ is multiplicity free. More precisely, each integer spin module appears once in its decomposition into a direct sum of irreducible $U_q(sl(2))$ -modules. Moreover, each element of $\mathcal{A}_{h,q}^c$ can be represented in a unique way as a sum of homogeneous elements such that each of them belongs to one of the highest components of $V^{\otimes i}$ (a proof of this fact is given in [A]).

We will treat the algebra $\mathcal{A}_{0,q}^c$ as a q -analogue of the commutative algebra $\text{Fun}(H)$ and call it *quantum hyperboloid* if $c \neq 0$ (being equipped with an involution it can be treated as the Podles quantum sphere ([P])) and *quantum cone* if $c = 0$. This is motivated by an analogy with the classical case. Since the $sl(2)$ -module $sl(2)^{\otimes 2}$ is multiplicity free we can introduce q -analogues I_{\pm} of symmetric and skew symmetric subspaces of $sl(2)^{\otimes 2}$ by setting, as in the classical case

$$I_+ = V_0 \oplus V_2 \quad I_- = V_1.$$

Let us emphasize that the corresponding algebras $A_{\pm} = T(sl(2))/\{I_{\mp}\}$ are flat deformations of their classical counterparts. Since the algebra $\mathcal{A}_{0,q}^c$ is a factor of the “ q -symmetric” algebra A_+ we consider it also as “ q -symmetric”. It is a particular case of the family of two parameters algebras $\mathcal{A}_{h,q}^c$ (where c is assumed to be fixed).

Remark 1 *In the case $n \geq 3$, the $sl(n)$ -module $sl(n)^{\otimes 2}$ is not multiplicity free. Consequently, it is not obvious what q -analogues of the symmetric and skew symmetric algebras of the space $sl(n)$ should be. However, as shown in [D], there exists a flat deformation of the algebra $\text{Sym}(sl(n))$. A way to construct these algebras in a more explicit way is suggested in [AG].*

For any simple Lie algebra g different from $sl(n)$ its tensor square $g^{\otimes 2}$ is multiplicity free. This implies that there exists natural q -analogues I_{\pm}^q of the subspaces $I_{\pm} \subset g^{\otimes 2}$. However, the q -algebras $A_{\pm}^q = T(V)/\{I_{\mp}^q\}$ are not flat deformations of their classical counterparts (cf. [G]).

3 Quantum anchor on quantum hyperboloid

In the present section we introduce the tangent space on the quantum hyperboloid and we equip it with a quantum anchor.

In the classical case, we can restate the identity (1) in a symbolic way as

$$(V \otimes V')_0 = 0 \tag{3}$$

where $V = \text{Span}(u, v, w)$, $V' = \text{Span}(U, V, W)$ and $(V \otimes V')_i$ designs the spin i $U(sl(2))$ -module in the tensor product $V \otimes V'$.

In order to define the tangent space on the quantum hyperboloid, we should consider the

identity (3) in the $U_q(sl(2))$ -module category. Denote V'^q the q -analogue of V' . Let $(\mathbf{V}^q \otimes V'^q)_0$ be the spin 0 $U_q(sl(2))$ -module and $\{(\mathbf{V}^q \otimes V'^q)_0\}$ be the left $\mathcal{A}_{0,q}^c$ -module generated by $(\mathbf{V}^q \otimes V'^q)_0$.

Definition 1 *The “left tangent space” on the quantum hyperboloid consider as a left $\mathcal{A}_{0,q}^c$ -module is defined as follows :*

$$T(H_q)_l = (\mathcal{A}_{0,q}^c \otimes V'^q) / \{(\mathbf{V}^q \otimes V'^q)_0\}. \quad (4)$$

In a similar way one can define the right tangent space $T(H_q)_r$ of the quantum hyperboloid considered as a right $\mathcal{A}_{0,q}^c$ -module.

Proposition 1 *([A], [AG])*

The $\mathcal{A}_{0,q}^c$ -module $T(H_q)$ is a flat deformation of its classical counterpart.

Now we discuss the problem of a suitable definition of a “quantum anchor” on the quantum hyperboloid.

For this, we need the notion of a braided Lie algebra. Following [DG], we define a braided Lie bracket

$$[\cdot, \cdot]_q : V^{\otimes 2} \longrightarrow V$$

as a no trivial $U_q(sl(2))$ -morphism (here V is spin 1 $U_q(sl(2))$ -module). By this request, the bracket is defined uniquely up to a factor. The product table of such a bracket is given for example in [DG]. Let us note that this construction is generalized for the $sl(n)$ $n > 2$ case, in [LS].

The space V equipped with such bracket will be denoted by :

$$sl(2)_q := (V, [\cdot, \cdot]_q)$$

and will be called a braided Lie algebra. The enveloping algebra of this braided Lie algebra can be defined similarly to that of $\mathcal{A}_{\hbar,q}^c$ but with ometting the first relation of (2). This enveloping algebra will be denoted $\mathcal{A}_{\hbar,q}$. We disregard here the problem of suitable relation between \hbar and the factor coming in the definition of $sl(2)_q$, cf. [LS].

Let us consider (left) q -adjoint operators associated to the elements of the space V . For example the operator U^q asociated to the generator u is defined by

$$\begin{aligned} U^q : V &\longrightarrow V \\ z &\longmapsto U^q z = ad^q u(z) = [u, z]_q, \quad U^q 1 = 0. \end{aligned}$$

In a similar way the operators V^q, W^q associated to the generators v, w respectively are well defined on the space V .

Proposition 2 *The following holds*

$$(q^3 + q) u W^q + v V^q + (q + q^{-1}) w U^q = 0. \quad (5)$$

Here all components are treated as operators acting from the space V to $\mathcal{A}_{0,q}^c$.

Throughout, we can assumed that the operators U^q, V^q, W^q realize a representation of the braided Lie algebra $sl(2)_q$ (i.e. they satisfy the defining relation of its enveloping algebra). If it is not the case we can get such operators by a proper rescaling

$$U^q \longrightarrow \lambda U^q, \quad V^q \longrightarrow \lambda V^q, \quad W^q \longrightarrow \lambda W^q \quad \lambda \in \mathbf{K}.$$

Let us remark that such a rescaling does not break the identity (5).

Definition 2 *We say that the $\mathcal{A}_{0,q}^c$ -module $T(H_q)$ is equipped with a structure of “left quantum anchor” if there exists an action*

$$\beta : T(H_q) \otimes \mathcal{A}_{0,q}^c \longrightarrow \mathcal{A}_{0,q}^c$$

such taht the operators corresponding to U^q, V^q, W^q realize a representation of the braided Lie algebra $sl(2)_q$ and the diagram

$$\begin{array}{ccc} \mathcal{A}_{0,q}^c \otimes T(H_q) \otimes \mathcal{A}_{0,q}^c & \longrightarrow & T(H_q) \otimes \mathcal{A}_{0,q}^c \\ \downarrow & & \downarrow \\ \mathcal{A}_{0,q}^c \otimes \mathcal{A}_{0,q}^c & \longrightarrow & \mathcal{A}_{0,q}^c \end{array}$$

is commutative. Here we suppose that the elements of $\mathcal{A}_{0,q}^c$ act on $\mathcal{A}_{0,q}^c$ (the low arrow) by the usual product. The vertical arrows are defined by means of β and the top one makes use of the $\mathcal{A}_{0,q}^c$ -module structure of $T(H_q)$.

In a similar way a notion of “right quantum anchor” can be defined.

Now we want to describe a quantum anchor in terms of the operators U^q, V^q, W^q . However, up to now the action of the operators U^q, V^q, W^q is well defined on the space V . To complet the construction of quantum anchor we should extend their action on the whole algebra $\mathcal{A}_{0,q}^c$ with propoities from definition 2. In the classical case, such an extension can be done via the Leibniz rule.

In the case under consideration such an extension is realize in $[A]$ (without any Leibniz rule). We represent here only the final result.

Theorem 1 *([A]) A quantum anchor exists and, moreover, it is unique if we impose the additional condition that the extented operators U^q, V^q, W^q send the highest component (V_i) of $V^{\otimes i}$ into itself.*

To finish this note, let us remark that the tangent space $T(H_q)$ is realize as an operator algebra on $\mathcal{A}_{0,q}^c$ and, moreover, the braided Lie algebra $sl(2)_q$ is embeded in it. This is the motivation of our definition of quantum anchor, inspite of the fact that we do not equip $T(H_q)$ with any structure fo Lie algebra (deformed or “braided”).

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